Chapter 5

Power Series

5.1 Definitions of power series at any point a and at $a = 0$

A power series is a type of series with terms involving a variable x. All terms of this series involve powers of x, and hence a power series can be thought of as an infinite polynomial. Power series are used to represent common functions and also to define new functions.

In this chapter we define power series and show how to determine when a power series converges and when it diverges. We also show how to represent certain functions using power series as well as Taylor series.

Definition 5.1: An infinite series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots . \qquad (5.1.1)$$

is called **a power series in** $(x - c)$ **or centered at** $x = c$, where $c_n's$ are coefficients of the series, x is a variable, and a is a constant.

If $a = 0$ then (5.1.1) becomes

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots . \qquad (5.1.2)$$

and is called a power series centered at 0 or $a = 0$.

Example 5.1: $\sum_{n=0}^{\infty} n!x^n = 1+x+2!x^2+3!x^3+\cdots$ is a power series centered at 0.

Example 5.2: $\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n} = 1 + \frac{(x-1)}{2} + \frac{(x-1)^2}{4} + \frac{(x-1)^3}{8} + \cdots$ is a power series centered at 1.

5.2 Convergence and Divergence of Power Series: Radius and Interval of Convergence

A power series in x can be viewed as a function of x. i.e, $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ with domain: the set of all x for which $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges.

Theorem 5.1: For a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, exactly one of the following is true.

1. The series converges only at $x = a$ (at its center),and diverges for all $x \neq a$.
2. The series converges absolutely for all x.
3. There exist a positive number R such that the series converges absolutely if $|x-a| < R$, and diverges if $|x-a| > R$. At the values of x where $|x-a| = R$, the series may converge or diverge.

The number in case (3) is called **the radius of convergence** of $\sum_{n=0}^{\infty} c_n(x-a)^n$. The set of all values of x for which the power series converges is called **the interval of convergence**.

The interval of convergence for a given power series takes one and only one of the forms: $(a-R, a+R)$, $(a-R, a+R]$, $[a-R, a+R)$ or $[a-R, a+R]$.

If $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges only at $x = a$, then $R = 0$ and the interval of convergence is a single point $\{a\}$ or $[a, a]$.

If $\sum_{n=0}^{\infty} c_n(x-a)^n$ for all x, then $R = \infty$ and the interval of convergence is the entire real number, $(-\infty, +\infty)$.

Activity I: For what values of x do the following power series converge? (Hint:

Use the generalized ratio test in chapter 4.)

$$(a)\ \sum_{n=0}^{\infty} n!x^n \qquad (b)\ \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (c)\ \sum_{n=1}^{\infty} \frac{(x-1)^n}{n} \qquad (d)\ \sum_{n=0}^{\infty} x^n.$$

Example 5.2: Find the radius and interval of convergence of the power series

$$(a)\ \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (b)\ \sum_{n=0}^{\infty} n!(x-2)^n \qquad (c)\ \sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2}.$$

Solution:

(a) • At $x = 0$ (the center), the series converges trivially.

• If $x \neq 0$, take $a_n = \frac{x^n}{n!}$ and $a_{n+1} = \frac{x^{n+1}}{(n+1)!}$, then applying the Generalized Ratio Test (GRT), we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}.\frac{n!}{x^n}\right| = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$$

Hence $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely for all x. Therefore, its radius of convergence is $R = \infty$, and its interval of convergence is $(-\infty, +\infty)$.

(b) • At $x = 2$, the series converges trivially.

• If $x \neq 2$, take $a_n = n!(x-2)^n$ and $a_{n+1} = (n+1)!(x-2)^{n+1}$, then by GRT, we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(x-2)^{n+1}}{n!\ (x-2)^n}\right| = |x-2| \lim_{n\to\infty}(n+1) = +\infty.$$

Hence $\sum_{n=0}^{\infty} n!(x-2)^n$ converges only at its center and diverges for all $x \neq 2$. Therefore, the radius of convergence is $R = 0$ and the interval of convergence is $[2, 2] = \{2\}$.

(c) • At $x = 5$, the series converges trivially.

• If $x \neq 5$, take $a_n = \frac{(x-5)^n}{n^2}$ and $a_{n+1} = \frac{(x-5)^{n+1}}{(n+1)^2}$, then by GRT, we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{(n+1)^2}.\frac{n^2}{(x-5)^n}\right| = |x-5| \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = |x-5|.$$

Hence $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2}$ converges absolutely if $|x-5| < 1$ with radius of convergence $R = 1$, and diverges if $|x-5| > 1$.

Test for endpoint convergence: $|x-5| < 1$ implies $4 < x < 6$

- When $x = 4$, then $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, which is convergent absolutely.
- When $x = 6$, then $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is also convergent.

Therefore the interval of convergence is $[4, 6]$.

Exercise 5.2: Find the radius and interval of convergence of the power series

$$(a) \ \sum_{n=0}^{\infty} \frac{x^n}{n} \qquad (b) \ \sum_{n=1}^{\infty} \frac{(x-2)^n}{2^{n+1}} \qquad (d) \ \sum_{n=0}^{\infty} \frac{n(2x+1)^n}{3^n}.$$

5.3 Algebraic Operation on Convergent Power Series

5.3.1 Addition of power series

Let $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ be two power series, then the two power series can be added and subtracted term by term just like series of constants. That is,

$$\sum_{n=0}^{\infty} a_n x^n \pm \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n = \sum_{n=0}^{\infty} c_n x^n.$$

The interval of convergence of the resulting sum or difference is the intersection of the intervals of convergence of the two original series.

Activity: Let $\sum_{n=0}^{\infty} x^n$ and $\sum_{n=0}^{\infty} (\frac{x}{2})^n$ be two power series, then find the interval of convergence of each power series and their sum.

5.3.2 Multiplication of power series

Two power series can be multiplied just as we multiply polynomials.

Theorem: If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converges absolutely for $|x| < R$ and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\Big(\sum_{n=0}^{\infty} a_n x^n\Big) . \Big(\sum_{n=0}^{\infty} b_n x^n\Big) = \sum_{n=0}^{\infty} c_n x^n.$$

Finding the general coefficient c_n in the product of two power series can be very tedious. Therefore, we often limit the computation of the product to the first few terms.

Example:

5.4 Representations of Functions as Power Series

In this section we deal with representing certain functions as power series either by manipulating geometric series or by differentiating or integrating of such a series. The reason for doing this is that it provides us with a way of integrating functions that do not have elementary antiderivatives, for solving certain differential equations, or for approximating functions by polynomials.

We start by recalling the geometric series

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{n=0}^{\infty} ar^n$$

which converges if $|r| < 1$ and diverges if $|r| \geq 1$. When it converges, its value is $\frac{a}{1-r}$.

Letting $a = 1$ and replacing r with x, we can express it as a function below.

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \quad \text{for} \quad |x| < 1. \tag{5.4.1}$$

Then the function on the left and the power series on the right produce the same result.

Example: Find a power series representation of :

$$(a)\ \ f(x)=\frac{1}{1-x^2}\qquad (b)\ \ f(x)=\frac{1}{1+x}\qquad (c)\ \ f(x)=\frac{x^2}{1-x^2}\qquad (d)\ \ f(x)=\frac{1}{4+2x}$$

Solution: (a) Replacing x by x^2 in (5.4.1), we get

$$\frac{1}{1-x^2}=1+x^2+(x^2)^2+(x^2)^3+\cdots=\sum_{n=0}^{\infty}(x^2)^n=\sum_{n=0}^{\infty}x^{2n}\ ,\qquad |x|<1.$$

(b) Replacing x by $-x$ in (5.4.1), we get

$$\frac{1}{1+x}=\frac{1}{1-(-x)}=1+(-x)+(-x)^2+(-x)^3+\cdots=\sum_{n=0}^{\infty}(-x)^n=\sum_{n=0}^{\infty}(-1)^n\ x^n,\ \ |x|<1$$

(c) Multiplying the series representation of $f(x)=\frac{1}{1-x^2}$ by x^2, we get

$$\frac{x^2}{1-x^2}=x^2\ \frac{1}{1-x^2}=x^2\sum_{n=0}^{\infty}x^{2n}=\sum_{n=0}^{\infty}x^{2n+2};\ \ |x|<1.$$

(d) Write the given function in the form of (5.4.1), and proceeding as above, we get

$$\frac{1}{4+2x}=\frac{1}{4(1-(\frac{-x}{2}))}=\frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{-x}{2}\right)^n=\frac{1}{2^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}x^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+2}}x^n.$$

This series converges if $|\frac{-x}{2}|<1$, which is the same as $|x|<2$, or $-2<x<2$.

Exercise:

5.4.1 Differentiation and Integration of Power Series

The domain of the function $f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n$ is the interval of convergence of the series. Differentiation and integration of functions represented by power series can be computed similarly as we can differentiate or integrate each terms for polynomial functions. This is called term-by-term differentiation and integration.

Theorem 5.4.1.1: If the power series $\sum_{n=0}^{\infty}c_n(x-a)^n$ has radius of convergence $R>0$, then the function f defined by

$$f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n=c_0+c_1(x-a)+c_2(x-a)^2+\cdots$$

is differentiable on the interval $(a-R, a+R)$ and

- $$f'(x) = \sum_{n=1}^{\infty} n\, c_n (x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots \quad (5.4.2)$$

- $$\int f(x)\, dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots \quad (5.4.3)$$

The radius of convergence of the series obtained by differentiating or integrating a power series is the same as that of the original power series , but the interval of convergence may differ.

Equations (5.4.2) and (5.4.3) above can be written in the form

$$\frac{d}{dx}\Big[\sum_{n=0}^{\infty} c_n (x-a)^n\Big] = \sum_{n=0}^{\infty} \frac{d}{dx}[c_n(x-a)^n] \quad \text{and}$$

$$\int \Big[\sum_{n=0}^{\infty} c_n (x-a)^n\Big]\, dx = \sum_{n=0}^{\infty} \int c_n (x-a)^n\, dx$$

Example: Find a power series representation and radius of convergence of the following functions :

$$(a)\ \ f(x) = \frac{1}{(1-x)^2} \qquad (b)\ \ f(x) = \ln(1+x) \qquad (c)\ \ f(x) = \tan^{-1}(x)$$

Solution:

(a) $\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right)$. Thus,

$$f(x) = \frac{1}{(1-x)^2} = \frac{d}{dx}\Big(\frac{1}{1-x}\Big) = \frac{d}{dx}\Big(\sum_{n=0}^{\infty} x^n\Big) = \sum_{n=1}^{\infty} n x^{n-1}, ; \quad |x| < 1.$$

$$(b)\ \ f(x) = \ln(1+x) = \int \frac{1}{1+x}\, dx = \int \Big(\sum_{n=0}^{\infty} (-1)^n\, x^n\Big)\, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{n+1}}{n+1},$$

$|x| < 1$.

To determine the value of C, we put $x = 0$ in the equation and we get $C = 0$.

Thus $f(x) = \ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n \; x^{n+1}}{n+1}$

(c) $f(x) = \tan^{-1}(x) = \int \frac{1}{1+x^2}\, dx = \int \Big(\sum_{n=0}^{\infty} (-1)^n \; x^{2n}\Big)\, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n \; x^{2n+1}}{2n+1}$,

$|x| < 1$.

When $x = 0$, $\tan^{-1}(0) = 0$ and hence $C = 0$. Therefore, the power series representation of $f(x) = \tan^{-1}(x)$ is $\sum_{n=0}^{\infty} \frac{(-1)^n \; x^{2n+1}}{2n+1}$.

Exercise:

5.5 Taylor Series; Taylor Polynomial and Application

In the preceding section we were able to find power series representation for a certain restricted class of functions. In this section we investigate more general cases for finding power series representations.

Suppose that $f(x)$ is any function that has a power series representation or expansion centered at $x = a$ that is valid for all x in an interval $(a-R, a+R)$ with $R > 0$:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots .$$

Then by theorem 5.4.1.1, we may differentiate the series term by term to obtain

$$\begin{aligned} f(x) &= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \\ f'(x) &= c_1 + 2\, c_2(x-a) + 3\, c_3(x-a)^2 + 4\, c_4(x-a)^3 + \cdots \\ f''(x) &= 2\, c_2 + 2\cdot 3\, c_3(x-a) + 3\cdot 4\, c_4(x-a)^2 + 4\cdot 5\, c_5(x-a)^3 + \cdots \\ &\vdots \\ f^{(k)}(x) &= k!\, c_k + (k+1)!\, c_{k+1}\,(x-a) + \cdots \end{aligned}$$

Setting $x = a$ in each of these series, we find that

$$f(a) = c_0,\; f'(a) = c_1,\; f''(a) = 2\, c_2,\; \cdots,\; f^{(k)}(a) = k!\, c_k,$$

and solving for c_k, we get the formula

$$c_k = \frac{f^{(k)}(a)}{k!}.$$

5.5.1 Taylor Series

Definition: Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series** generated by f at $x = a$ is:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots .$$

If $a = 0$, then the Taylor series $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ is called the **Maclaurin series** of f.

Example: Find the Taylor Series for the functions

(a) $f(x) = e^x$ at $a = 1$ and $a = 0$;

(b) $f(x) = \frac{1}{x}$ at $a = 2$;

(c) $f(x) = \sin\ x$ at $a = 0$

(d) $f(x) = \ln(x+1)$at $a = 0$

Solution: (a) • When $a = 1$, we have,

$$\begin{aligned} f(x) &= e^x\ , & f(1) &= e\ , \\ f'(x) &= e^x\ , & f'(1) &= e\ , \\ f''(x) &= e^x\ , & f''(1) &= e\ , \\ &\vdots & &\vdots \\ f^{(n)}(x) &= e^x\ , & f^{(n)}(1) &= e \end{aligned}$$

Therefore,

$$f(x) = e^x = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots + \frac{f^{(n)}}{n!}(x-1)^n + \cdots$$

Hence, the Taylor series of $f(x) = e^x$ at $a = 1$ is

$$e^x = e + \frac{e}{1!}(x-1) + \frac{e}{2!}(x-1)^2 + \cdots + \frac{e}{n!}(x-1)^n + \cdots = \sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n$$

• When $a = 0$, $f^{(n)}(0) = e^0 = 1$ for all n, and thus $f(0) = f'(0) = f''(0) = \cdots = f^{(n)}(0) = 1$, with coefficient $c_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$. Therefore, the resulting series is the Maclaurine series, which is given by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

(b)

$$\begin{aligned}
f(x) &= \frac{1}{x}, & f(2) &= \frac{1}{2}, \\
f'(x) &= \frac{-1}{x^2}, & f'(2) &= \frac{-1}{2^2}, \\
f''(x) &= \frac{2}{x^3} = \frac{2!}{x^3}, & f''(2) &= \frac{2}{2^3} = \frac{2!}{2^3}, \\
f'''(x) &= \frac{-6}{x^4} = \frac{-3!}{x^4}, & f'''(2) &= \frac{-6}{2^4} = \frac{-3!}{2^4}, \\
&\vdots & &\vdots \\
f^{(n)}(x) &= \frac{(-1)^n n!}{x^{(n+1)}}, & f^{(n)}(2) &= \frac{(-1)^n n!}{2^{(n+1)}},
\end{aligned}$$

Therefore, the Taylor series of $f(x) = \frac{1}{x}$ is

$$\begin{aligned}
& f(2) + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots \\
= \ & \frac{1}{2} - \frac{1}{2^2 \cdot 1!}(x-2) + \frac{2!}{2^3 \cdot 2!}(x-2)^2 - \frac{3!}{2^4 \cdot 3!}(x-2)^3 + \cdots + \frac{(-1)^n n!}{2^{(n+1)} n!}(x-2)^n + \cdots \\
= \ & \frac{1}{2} - \frac{1}{2^2}(x-2) + \frac{1}{2^3}(x-2)^2 - \frac{1}{2^4}(x-2)^3 + \cdots + \frac{(-1)^n}{2^{(n+1)}}(x-2)^n + \cdots \\
= \ & \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^n.
\end{aligned}$$

(c)

(d)

Remark: If f can be represented as a power series about a, or f has derivatives of all orders on an interval containing a as center, then f is equal to the sum of its Taylor series. That is

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}\,(x-a)^k = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots .$$

Activity: Does there exist functions that are not equal to the sum of their Taylor series?

Exercise: Find the Taylor series of:

5.5.2 Taylor Polynomials and Application

Definition: Let f be a function such that the n^{th} derivative, $f(x)^{(n)}$ exists at a in some interval containing a as interior point. Then the polynomial:

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^2,$$

or $\quad p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}\,(x-a)^k$is called the$n^{th}$degree Taylor's polynomials of$f$at$a$.

Example: Find the Taylor polynomial of

$$a) f(x) = e^x \text{ at } a = 2, \qquad b)\ f(x) = \frac{1}{1-x} \text{ at } a = -1$$

Solution: a)

$$\begin{aligned} f(x) &= e^x\,, & f(2) &= e^2\,, \\ f'(x) &= e^x\,, & f'(2) &= e^2\,, \\ f''(x) &= e^x\,, & f''(2) &= e^2\,, \\ &\vdots & &\vdots \\ f^{(n)}(x) &= e^x\,, & f^{(n)}(2) &= e^2 \end{aligned}$$

Therefore,

$$\begin{aligned} p_n(x) &= f(2) + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n \\ &= \sum_{k=0}^{n} \frac{e^2}{n!}(x-2)^k \end{aligned}$$

b)

$$\begin{aligned} f(x) &= \frac{1}{1-x}\,, & f(-1) &= \frac{1}{2}\,, \\ f'(x) &= \frac{1}{(1-x)^2}\,, & f'(-1) &= \frac{1!}{2^2}\,, \\ f''(x) &= \frac{2}{(1-x)^3}\,, & f''(-1) &= \frac{2!}{2^3}\,, \\ f^{(3)}(x) &= \frac{6}{(1-x)^4}\,, & f^{(3)}(-1) &= \frac{3!}{2^4}\,, \\ &\vdots & &\vdots \\ f^{(n)}(x) &= \frac{n!}{(1-x)^{n+1}}\,, & f^{(n)}(-1) &= \frac{n!}{2^{n+1}} \end{aligned}$$

Therefore,

$$\begin{aligned} p_n(x) &= f(-1) + \frac{f'(-1)}{1!}(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \cdots + \frac{f^{(n)}(-1)}{n!}(x+1)^n \\ &= \frac{1}{2} + \frac{1}{2^2}(x+1) + \frac{1}{2^3}(x+1)^2 + \cdots + \frac{1}{2^{n+1}}(x+1)^n \\ &= \sum_{k=0}^{n} \frac{1}{2^{k+1}}(x+1)^k \end{aligned}$$

Exercise: Find a formula for the n^{th} Taylor polynomial of the functions

$a) f(x) = \sqrt{x}$ at $a = 4$, $\quad b)\ f(x) = \cos x$ at $a = 0$, $\quad c)\ f(x) = \ln(1+x)$ at $a = 0$

Theorem (Taylor's Theorem): If f is differentiable $(n+1)$ times on an interval I containing a and x, and if $p_n(x)$ is the Taylor polynomial of degree n for f about $x = a$, then

$$f(x) = p_n(x) + R_n(x) \qquad \textbf{(Taylor formula with remainder)},$$

where the remainder (error) term $R_n(x)$ is called **Lagrange remainder** and is given by

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \text{ for some } c \text{ between } x \text{ and } a.$$

If $R_n(x) \to 0$ *as* $n \to \infty$ on I, we say that the Taylor series generated by f at $x = a$ *converges* to $f(x)$ on I, and we write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Theorem (The remainder estimation theorem): If $|f^{(n+1)}(x)| \leq M$ for all x in I, in the above theorem, then

$$|R_n(x)| = \frac{M}{(n+1)!}|x-a|^{n+1} \text{ for all} x \in I.$$

Example 1: Estimate $f(x) = \sin x$ by a 4^{th} degree polynomial in x at $a = 0$ if $0 \leq x \leq 0.2$.

Solution:

$$\begin{aligned} f(x) &= \sin x\ , & f(0) &= 0\ , \\ f'(x) &= \cos x\ , & f'(0) &= 1\ , \\ f''(x) &= -\sin x\ , & f''(0) &= 0\ , \\ f'''(x) &= -\cos x\ , & f'''(0) &= -1\ , \\ f^{(4)}(x) &= \sin x\ , & f^{(4)}(0) &= 0;\ , \\ f^{(5)}(x) &= \sin x\ , & f^{(5)}(c) &= \cos c;\ , \end{aligned}$$

Therefore,

$$p_{n=4}(x) = x - \frac{x^3}{6},\ \text{and } |R_4(x)| = \frac{\cos c}{5!}x^5 \leq \frac{1}{120}x^5, \text{since } |f^{(5)}(c)| = |\cos c| \leq 1.$$

Now for $0 \leq c \leq 0.2$, take $x = 0.2$ and $|R_4(x)| \leq \frac{1}{120}(0.2)^5 \approx 2.\dot{6} \times 10^{-6} < 3 \times 10^{-6}$. Hence, in the given range $f(x) = \sin x$ is estimated by

$$f(x) = \sin x = x - \frac{x^3}{6} \pm 0.000003.$$

Example 2 Approximate $\sqrt[3]{e}$ to an accuracy of **five** decimal place.

Solution: By Taylor formula $f(x) = P_n(x) + R_n(x)$. That is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + R_n(x), \text{where } R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

Since $f(x) = e^x$, $\sqrt[3]{e} = e^{\frac{1}{3}} = f(\frac{1}{3})$. So we want to approximate $f(\frac{1}{3})$.

Consider $P_n(x)$ is an approximation, then the error will be $|f(1/3) - P_n(1/3)| = |R_n(1/3)|$. Consequently, if we find an n such $|R_n(x)| < 10^{-5} = 0.00001$. Then $P_n(1/3)$ will be the desired approximation.

Now by Taylor Theorem, there exist a number c between 0 and 1/3 such that $R_n(1/3) = \frac{f^{(n+1)}(c)}{(n+1)!}(\frac{1}{3})^{n+1}$.

Since $f^{(n+1)}(c) = e^c$ and $e^{\frac{1}{3}} < 2$, $R_n(1/3) = \frac{e^c}{(n+1)!}(\frac{1}{3})^{n+1} < \frac{2}{(n+1)!}(\frac{1}{3})^{n+1}$

By Computing $\frac{2}{(n+1)!}(\frac{1}{3})^{n+1}$ for $n = 1, 2, \cdots, 5$, we get that $|R_n(1/3)| < 10^{-5}$ if $n \geq 5$. i.e $|R_5(1/3)| < \frac{2}{6!3^6} = \frac{1}{(360)(729)} < 10^{-5}$.

Thus $P_5(\frac{1}{3})$ is the desired approximation to $\sqrt[3]{e}$.

Therefore

$$P_5(\frac{1}{3}) = 1 + \frac{1}{3} + \frac{1}{2!}(\frac{1}{3})^2 + \frac{1}{3!}(\frac{1}{3})^3 + \frac{1}{4!}(\frac{1}{3})^4 + \frac{1}{5!}(\frac{1}{3})^5 = \frac{5087}{3645} \approx 1.39561.$$

Hence, we conclude that 1.39561 approximate $\sqrt[3]{e}$ with an error less than 10^{-5}.

Example 3: Evaluate the integrals

$$a) \int e^{-x^2} dx \text{ as an infinite series}$$

$$b) \int_0^1 e^{-x^2} dx \text{ correct to within an error of } 0.001$$

Solution a) Using the Maclaurin series of e^x and substituting x by $-x^2$ we get

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

Integrating term by term

$$\begin{aligned} \int e^{-x^2} dx &= \int \Big(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\Big) dx \\ &= C + x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots \\ &= C + \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{(2n+1) \, n!} \end{aligned}$$

b) By fundamental theorem of calculus,

$$\begin{aligned} \int_0^1 e^{-x^2} dx &= \Big[C + x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots\Big]\Big|_0^1 \\ &= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots \\ &\approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \\ &\approx 0.7475 \end{aligned}$$

The Taylor Estimation Theorem shows that the error involved in this approximation is less than $\frac{1}{11 \cdot 5!} = \frac{1}{1320} < 0.001$.

Example 4 Determine the Taylor series of $\int \frac{\sin x}{x}\, dx$

Solution Using the Taylor series $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!}$ about $x = 0$,

$$\frac{\sin x}{x} = \frac{1}{x}\, \sin x = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n+1)!}.$$

Thus,

$$\int \frac{\sin x}{x}\, dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n+1)!}\, dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n\, x^{2n}}{(2n+1)!}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)(2n+1)!} + C.$$

Example 5 Approximate $\int_0^1 \sin(x^2)\, dx$ to four decimal place **Solution** Using the Taylor series representation

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

Thus

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots .$$

Applying term by term integration and the fundamental theorem of calculus, we get

$$\begin{aligned}
\int_0^1 \sin(x^2)\, dx &= \left[\frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \cdots \right] \Bigg|_0^1 \\
&= \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75600} + \cdots \\
&\approx 0.3103, \quad \text{which is the sum of the first three terms.}
\end{aligned}$$

Exercise: 1. Approximate e with an error of less than 10^{-6}.

2. Use the Maclaurin series for $\sin x$ to approximate $\sin 3^o$ to five decimal place accuracy.

3. Approximate $\ln 2$ with an error less than 0.01.

4.

5.

6.

Binomial Series

Binomial series is a Taylor series with many applications. Some of the application of Binomial series is to approximate power or root functions of the form $f(x) = (1+x)^m$, where m is any constant.

The Maclaurin series generated by $f(x) = (1+x)^m$ is

$$\begin{aligned} f(x) = (1+x)^m &= 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots \\ &\quad \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}x^k + \cdots \\ &= \sum_{k=0}^{\infty} \binom{m}{k} x^k \end{aligned}$$

This series is called binomial series, which converges absolutely for $|x| < 1$.

$\binom{m}{k}$ is called binomial coefficient, and defined by $\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}$.

In particular, $\binom{m}{0} = 1, \binom{m}{1} = m, \binom{m}{2} = \frac{m(m-1)}{2!}$

Example 1: Find the Maclaurin series of $f(x) = (1+x)^{-1}$.

Solution: $m = -1; \binom{-1}{0} = 1; \binom{-1}{1} = -1; \binom{-1}{2} = \frac{-1(-1-1)}{2!} = 1; \cdots; \binom{-1}{k} = \frac{-1(-2)(-3)\cdots(-k)}{k!} = (-1)^k \binom{k!}{k!} = (-1)^k$

Therefore, $(1+x)^{-1} = \sum_{k=0}^{\infty} (-1)^k x^k = 1 - x + x^2 - x^3 + \cdots + (-1)^k x^k + \cdots$.

Example 2: Find the Maclaurin series of $f(x) = (1+x)^{\frac{1}{2}}$.

Solution; $m = 1/2; \binom{1/2}{0} = 1; \binom{1/2}{1} = 1/2; \binom{1/2}{2} = -1/8; \binom{1/2}{3} = 1/16; \binom{1/2}{4} = -15/384; \cdots; \binom{1/2}{k} = (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdots (2k-3)}{2^k k!}$.

Therfore,

$$\begin{aligned}(1+x)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} x^k &= 1 + \frac{1}{2}x + \sum_{k=2}^{\infty} (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdots (2k-3)}{2^k k!} x^k \\ &= 1 + \frac{1}{2}x + \sum_{k=2}^{\infty} (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdots (2k-3)}{2 \cdot 4 \cdot \cdot 6(2k)} x^k.\end{aligned}$$

Chapter Summary: